PMM U.S.S.R.,Vol.45,pp.374-379
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0021-8928/82/3 0374 $7.50/0
UDC 531.36

ON THE FEASIBILITY OF STABILIZING STEADY MOTIONS OF SYSTEMS WITH PSEUDO-IGNORABLE COORDINATES*

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A class of systems with pseudo-ignorable coordinates is introduced. Set P of possible steady motions is described, and the problem of steady motion stabilization is qualitatively analyzed. A similar problem was considered in /1,2/ from the point of view of the general theory of controlability. Information on set P, an instrument of invariant manifolds of linear systems, and the Kelvin-Chetaev theorems provide in a number of cases the means for a simple and effective solution of this problem.

1. The set of steady motions. Let among the generalized coordinates q_i $(i = 1, \ldots, n)$ of a mechanical system with steady holonomic constraints there be coordinates q_j $(j = r+1, \ldots, n; r < n)$ which do not explicitly appear in the expression for the system kinetic energy T. The forces acting on the system are assumed independent of such coordinates which are called below pseudo-ignorable. The remaining q_i $(i = 1, \ldots, r)$ are position coordinates. We shall use matrix notation, viz. q for a column matrix consisting of position coordinates, and $\dot q$, ω for column matrices of position and pseudo-ignorable velocities.

We assume the input system to be free of gyroscopic constraints, i.e.

$$2T = \dot q^T A(q) \dot q + \omega^T B(q) \omega$$

where A, B are positive definite matrices whose coefficients are independent of position coordinates.

Let us further assume that the generalized forces that correspond to position coordinates are specified and represent the sum of potential and dissipative forces

$$Q_i = dU/dq_i + Q_{id}$$

The generalized forces F_j which correspond to pseudo-ignorable coordinates are taken as the control forces subject to selection.

Let us assume that under certain initial conditions the following stabilized motions of the system are possible:

$$q(t) = q_0 = \text{const}, \quad \omega(t) = \omega_0 = \text{const}$$

For the determination of q_0, ω_0 we have the equations

$$-\frac{\partial u}{\partial q_i} - \frac{1}{2} \frac{\partial}{\partial q_i} \omega^T B \omega = 0, \quad F_j = 0 \tag{1.1}$$

One of the aims of the selection of control forces F_j is generallly to ensure the fulfillment of conditions of existence of steady motions at certain specified pseudo-ignorable velocities. This is evidently not possible for all systems, but only for those for which Eqs. (1.1) have at least one solution for q at a given ω_0. In applied problems ω_0 is, as a rule, assumed to belong to some domain Ω_0 of space $\Omega = \{\omega\}$. Any point of the domain Ω_0 can be chosen as representing the working mode.

The above assumptions are sufficient for analyzing Eqs.(1.1) as equations in q with parameters ω. In an n-dimensional space of variables q, ω these equations define the set P of possible steady motions of the system. Generally this set consists of a denumerable number of components P_e that can be represented in the form of single-valued functions

$$q = f_e(\omega), \quad e = 1, 2, \ldots$$

with the domain Ω_e of their determination is of dimension $n-r$ or smaller. Only those components of set P for which $\dim \Omega_e = n-r$ are taken into account in the subsequent analysis.

Many objects have components of set P for which functions f_e are independent of components of vector ω. This happens when the following equalities are satisfied on the hyperplane $q = q_0$:

*Prikl.Matem.Mekhan.,45,No.3,512-520,1981

$$dU/dq_i = dB_{kj}/dq_i = 0 \tag{1.2}$$

where B_{kj} are the coefficients of matrix B. We call such steady motions trivial, and the remaining, significant. The domain of determination of the component of set P, which consists of stable trivial motions, obviously coincide with the whole space Ω.

Individual components of set P may intersect at points of some set of smaller dimension L. Branching of solutions of the system of Eqs.(1.1) occurs at points of set L, and its Jacobian $\det W$ must necessarily vanish. A large number of mechanical system has the property that the condition of the Jacobian becoming zero is satisfied only at branching points, and not at any other points of set P. This enables us to introduce in the analysis some supplementary characteristic.

We denote by ν the number of negative eigenvalues of matrix W. Let us assume that the component P_e of set P is separated by "line" L in two (or more) coherent parts P_{e1} and P_{e2}. Since at least one of matrix W eigenvalues vanishes on L, it well may prove that the quantity ν assumes different values in parts P_{e1} and P_{e2}, while remaining the same at all points of one part. It is therefore advisable to distinguish the component of set P not only by the criterion of function f_e single-valuedness but, also, by the respective value of parameter ν which will be called below the index. (The introduced here concept of the index is an extension of that of Poincaré's degree of instability used in the case of special selection of control forces.) We call a connected single-valued component of set P the leaf P_e, if at all points of that component the index ν is constant, and none of its internal points are branching points of solution of Eqs.(1.1).

The set of leaves P_e can be considered to be a geometrical characteristic of the input system. The definition of this image must contain a list of leaves with indication of respective indices. For any selection of control forces F_j certain points of set P correspond to possible steady motions. It will subsequently become clear that the separation of the set in leaves is useful for the qualitative investigation of the stabilization problem and for the construction of stabilizing actions.

Note also that by far not all points of set P at which condition $\det W = 0$ are satisfied (singular points) have to be branching points. The question of the number of solutions in the singular points neighborhood requires further analysis (see, e.g., /3/).

Example 1. Let us describe the set of possible stable motions of a physical pendulum whose horizontal swinging axis OO' can turn about the vertical axis NN'. Some interesting properties of motions of such system were noted in /4,5/. The system has two degrees of freedom: turning of the swing axis by angle ψ (pseudo-ignorable coordinate) around of axis NN' and the turn of the pendulum body about the swing axis by angle ϑ (position coordinate). For simplicity we assume that axes OO' and NN' intersect at point O, and that axes OO' and OG (G is the body center of mass) are the principal axes of the body ellipsoid of inertia relative to point O. On these assumptions

$$2T = I_1\vartheta^{\cdot 2} + (I_2 \sin^2\vartheta + I_3\cos^2\vartheta)\omega^2$$
$$U = mga\cos\vartheta, \quad a = |OG|$$

where $\omega = \psi^{\cdot}$, m is the mass, and I_1, I_2, I_3 the respective moments of inertia of the body.

The system of Eqs.(1.1) is transformed into the single equation

$$mga\sin\vartheta - (I_2 - I_3)\,\omega^2 \sin\vartheta\cos\vartheta = 0$$

which obviously has branches of trivial solutions

$$\vartheta_k = \pi k, \quad k = 0, \pm 1, \ldots$$

and branches of significant solutions

$$\vartheta_* = \operatorname{Arccos}\frac{mga}{(I_2 - I_3)\,\omega^2}$$

which intersect with respective branch of trivial solutions when $\omega = \omega_*$, where $mga = \omega_*^2 |I_2 - I_3|$.

We restrict our considerations to that part of set P for which $0 \leqslant \vartheta \leqslant \pi$. It contains four leaves. When $I_2 > I_3$ the set of trivial motions $\vartheta = 0$ consisits of two leaves: P_1 $(0 \leqslant \omega < \omega_*)$ and P_2 $(\omega > \omega_*)$ with indices $\nu_1 = 0$ and $\nu_2 = 1$, respectively. There is also leaf P_3 of significant motions for $\omega > \omega_*$, with index $\nu_3 = 0$, and leaf P_4 of trivial motions $\vartheta = \pi$ with index $\nu_4 = 1$.

When $I_2 < I_3$ we have leaf P_1 of motions $\vartheta = 0$ $(\nu_1 = 0)$, motions $\vartheta = \pi$ which form two leaves P_2 $(\omega < \omega_*, \nu_2 = 1)$ and P_4 $(\omega > \omega_*, \nu_4 = 0)$. Leaf P_3 of significant motion has $\nu_3 = 1$ as the index.

The control force F is provided by the moment generated by the mechanism that turns the axis OO'.

Example 2. Let us consider a heavy gyroscope in a perfect universal joint with a vertical axis of rotation of the external gimbal ring /6,7/. In that case Eq.(1.1) is of the form

$$[(I_{3*} - I_{2*})\omega_2^2 \cos\vartheta + I_3\omega_2\omega_3 + mga] \sin\vartheta = 0 \tag{1.3}$$
$$I_{2*} = I_1 + J_2, \quad I_{3*} = I_3 + J_3$$

where the pseudo-ignorable velocities ω_3, ω_2 are angular velocities of the gyrostat proper rotation and precession, respectively, the position coordinate $\vartheta\,(0 \leqslant \vartheta \leqslant \pi)$ is the nutation angle, I_1, I_3, J_2, J_3 are, respectively, the moments of inertia of the rotor and inner gimbal ring, m is the gyroscope mass, and a the distance between the gyroscope center of mass and the suspension center.

Equation (1.3) has evidently two trivial solutions: $\vartheta_1 = 0$ and $\vartheta_2 = \pi$. When $I_{3*} \neq I_{2*}$ there are moreover in the parameter plane ω_2, ω_3 two domains Ω_7 and Ω_8 whose points correspond to supplementary solutions of Eq.(1.3)

$$\vartheta_{3,4} = \arccos \frac{mga + I_3\omega_2\omega_3}{\omega_2^2 (I_{2*} - I_{3*})}$$

The surfaces $\vartheta_{3,4}(\omega_2, \omega_3)$ intersect the plane $\vartheta = 0$ along the branches of the hyperbola

$$mga + I_3\omega_2\omega_3 + (I_{3*} - I_{2*})\omega_2^2 = 0 \tag{1.4}$$

and the plane $\vartheta = \pi$ along the branches of hyperbola

$$mga + I_3\omega_2\omega_3 + (I_{2*} - I_{3*})\omega_2^2 = 0 \tag{1.5}$$

Both hyperbolas constitute a set of branching lines L.

Having obtained the sign of the Jacobian of Eq.(1.3), we determine the indices of leaves of set P.

The plane $\vartheta = 0$ consists of three leaves: P_1 which is the part of the plane comprised between branches of the hyperbola (1.4) with index $\nu_1 = 0, P_2$, and $P_3\,(\nu_2 = \nu_3 = 1)$ the parts lying outside of hyperbola (1.4). The plane $\vartheta = \pi$ also consists of three leaves: $P_4\,(\nu_4 = 1)$ which are the parts of the plane inside hyperbola (1.5), P_5, and $P_6\,(\nu_5 = \nu_6 = 0)$ parts of the plane outside hyperbola (1.5). The signs of leaves P_7, P_8 of significant motions are determined by the sign of remainder $I_{3*} - I_{2*}$. If $I_{2*} < I_{3*}$, then $\nu_7 = \nu_8 = 1$ and $\Omega_7 = \Omega_1 \cap \Omega_5$, $\Omega_8 = \Omega_1 \cap \Omega_6$, and when $I_{2*} > I_{3*}$ we have $\nu_7 = \nu_8 = 0$ and $\Omega_7 = \Omega_2 \cap \Omega_5$, $\Omega_8 = \Omega_2 \cap \Omega_6$.

In the problem considered the control forces can be generated by: moment F_3 of the motor fitted to the inner gimbal ring and driving the gyroscope, and moment F_2 of the motor driving the outer gimbal ring.

2. First approximation equations. Let us consider the feasibility of stabilizing steady motions belonging to the various leaves of set P. As in /1,2/ we shall analyze first approximation equations, for which we linearize Lagrange equations in the neighborhood of motion q_0, ω_0. Introducing deviations $x = q - q_0, \eta = \omega - \omega_0$, we obtain

$$A_{i0}x^{\cdot\cdot} + D_{i0}x^{\cdot} + W_{i0}x - \left[\frac{\partial}{\partial q_i}(\omega^T B)\right]_0 \eta = 0 \tag{2.1}$$
$$x^{\cdot T}\left[\frac{\partial}{\partial q}(B_j\omega)\right]_0 + B_{j0}\eta^{\cdot} = K_j\eta + M_j x + N_j x^{\cdot}$$

where D_i is the i-th row of matrix D of the linear part of dissipative force Q_{id}, A_i, B_j, W_i are rows of matrices A, B, W, and K_j, M_j, N_j are rows of the respective matrices K, M, N of linear control forces. The zero subscript shows that the respective quantity is calculated for $q = q_0$, $\omega = \omega_0$. We assume that $\det D \neq 0$.

From the point of view the structure of forces sysyem (2.1) can be considered as the result of imposing on the two independent subsystems /8/

$$A_{i0}x^{\cdot\cdot} + W_{i0}x + D_{i0}x^{\cdot} = 0, \quad B_{j0}\eta^{\cdot} = 0 \tag{2.2}$$

of additional forces. This device enables us to obtain certain results on the basis of the Kelvin–Chetaev (Thomson and Tait /9/) theorems.

Note that the first subsystem is subjected to potential and dissipative forces. Its zero approximation is asymptotically stable, if the stable motion ω_0, q_0 belongs to the leaf with index $(\nu_e = 0)$, and is unstable if $\nu_e \neq 0$.

Equations (2.1) enable us to solve the problem of stabilization of steady trivial motions, when conditions (1.2) are satisfied and the first subsystems in (2.1) and (2.2) are the same. This enables us to make the following statements.

Statement 1. Any trivial stable motion that belongs to leaf P_e with index $\nu_e \neq 0$ cannot be stabilized by any linear control forces.

Statement 2. If a trivial stable motion belongs to a leaf with zero index, then for its stabilization it is sufficient that matrix $-K-K^T$ is positive definite.

Note that the property of amenability to stabilization or nonstabilization established on the basis of the linear equations (2.1) are also retained by virtue of complete equations of motion.

Example 1. Let us continue the analysis of motions of a physical pendulum. Statement 1 enables us to establish the impossibility of stabilizing steady motions belonging to leaves P_2, P_4 in the case of $I_2 > I_3$, and those belonging to P_2 in the case of $I_3 > I_2$. For the stabilization of remaining leaves of trivial steady motions it is sufficient that $F = k(\omega - \omega_0)$ $(k < 0)$.

Example 2. In the case of a gyroscope in universal joint we find that no linear forces F_2, F_3 can stabilize steady motions that belong to leaves P_2, P_3, P_4. Steady motions on leaves P_1, P_5, P_6 can be stabilized by, for example, using the following control: $F_j = k_j(\omega_j - \omega_{j0})$ $(k_j < 0)$, $j = 2, 3$.

3. Stabilization of significant steady motion. Let the motion mode $q = q_0$, $\omega = \omega_0$ belong to the leaf of significant steady motions. The problem of selecting the control forces structure can, also in this case, be reduced to the analysis of two independent subsystems by using the following method.

Let us stipulate that system (2.1) must have the invariant manifold

$$\eta + Hx + Gx^{\cdot} = 0 \tag{3.1}$$

For this it is sufficient that system

$$y^{\cdot} = -\gamma y, \quad y = B\eta + BHx + BGx^{\cdot} \tag{3.2}$$

where γ is an arbitrary matrix, is identically satisfied by virtue of Eqs.(2.1). The selection of matrices H, G, γ evidently uniquely determine the coefficients of matrices K, M, N of control forces.

Let us also stipulate that matrix γ must be symmetric and positive definite. The manifold (3.1) is then asymptotically stable.

The first of Eqs.(2.1) in the manifold (3.1) is a linear system of the over-all state under the action of total structure forces. Its dimension is, however, determined only by the number of position coordinates, which facilitates the analysis of its zero solution stability. The question of the investigated mode of motion is resolved on the basis of the following obvious statement.

Statement 3. If it is possible to select such matrices H and G that the subsystem of the first of Eqs.(2.1) in the manifold (3.1) has an asymptotically stable zero solution, the steady motion $q = q_0$, $\omega = \omega_0$ can be stabilized.

Statement 3 is only a particular form of the theorems on stabilization feasibility formulated in /1,2/ for the general case of gyroscopically constrained systems. The method of linear mechanical systems stabilization expounded in /9/ can also be used for devising stabilizing effects.

The problem of devising stabilizing effects is simplified if only matrices

$$G = 0, \quad H = \Gamma \left[\frac{\partial}{\partial q}(B\omega)\right]_0$$

where Γ is some symmetric positive definite matrix, are used. In that case the subsystem in the first of Eqs.(2.1) in the manifold (3.1) is of the form

$$A_{i0}x^{\cdot\cdot} + D_{i0}x^{\cdot} + W_{i0}x + W_{i0}'x = 0 \tag{3.3}$$
$$W_{i0}' = \left[\frac{\partial}{\partial q_i}(\omega^T B)\right]_0 \Gamma \left[\frac{\partial}{\partial q}(B\omega)\right]_0$$

System (3.3) differs from system (2.2) by the presence of additional potential forces with matrix W_0' formed of rows W_{i0}'.

If the steady motion belongs to the leaf with zero index, the asymptotic stability of the zero solution of system (3.3) is attained even when $\Gamma = 0$. Note also that matrix W_0' is nonnegative. It is, therefore, possible to expect stabilization of steady motions belonging to certain of the P_e leaves for which $\nu_e \neq 0$. The feasibility of such stabilization is formulated more precisely by the following statement.

Statement 4. Let $x^{(1)},\ldots, x^{(\nu_e)}$ be the eigen vectors of matrix W_0 that correspond to its negative eigenvalues, and m be the dimension of the linear subspace X

$$\left[\frac{\partial}{\partial q}(B\omega)\right]_0 x=0 \quad (m \geqslant 2r-n)$$

Then to stabilize a stable motion by the method considered here it is necessary that $\nu_e \leqslant r-m$ and sufficient that vectors $x^{(1)},\ldots, x^{(\nu_e)}$ do not belong to the subspace X.

When the conditions of Statement 4 are satisfied, it is possible to select matrix Γ (for example, in the form of a diagonal matrix) such that matrix W_0+W_0' of potential forces in the system of Eqs.(3.3) becomes positive definite. This ensures stability of the zero solution of system (3.3).

Having selected matrix Γ, we calculate matrices K, M, N using formulas

$$K=-\gamma B_0, \quad M=-\gamma B_0\Gamma\left[\frac{\partial}{\partial q}(B\omega)\right]_0, \quad N=\left[\frac{\partial}{\partial q}(B\omega)\right]_0-B_0\Gamma\left[\frac{\partial}{\partial q}(B\omega)\right]_0$$

Let us list the constraints that must be satisfied by the coefficients of matrices K, M, N of the control forces that solve the problem of stabilization. The first group of constraints consists of positive definiteness of matrix γ. The second group consists of conditions of positive definiteness of matrix W_0+W_0'. The arbitrariness of the selection of matrix γ, evidently, gives a fairly wide freedom in the selection of matrix K, M. In particular, the coefficients of matrix γ (respectively of matrices K, M) can be chosen arbitrarily small.

Differences in stability conditions for subsystem (3.3) were indicated in /4,10/ in the case of $\Gamma=0$ and $\Gamma=B^{-1}$.

Example 1. Consider a physical pendulum with a single position coordinate. In the case of steady motion on leaf P_3 we have

$$\partial B/\partial\vartheta=(I_3-I_2)\sin 2\vartheta\neq 0$$

Hence the linear subspace X degenerates into point $x=0$. If $I_2>I_3$, then $\nu_3=0$ and there is virtually no question of stabilizing a steady motion on leaf P_3. When $I_3>I_2$, we have $\nu_3=1$ but any steady motion belonging to that leaf, can be stabilized by selecting a reasonably large number Γ. Since

$$W'={}^1/_4\Gamma(I_3-I_2)^2\omega^2\sin^2 2\vartheta$$

condition $W_0+W_0'>0$, after appropriate transformations, assumes the form

$$\Gamma>(I_3-I_2)\omega_0^4\sin^2\vartheta_*(\omega_0)(mga)^{-1}$$

At high angular velocities the quantity Γ required for stabilization is of order ω_0^4. If the coefficient of inverse constraint with respect to pseudo-ignorable velocity (see (3.4)) and with respect to the position coordinate (K and M) can be reduced by selecting a small γ, the coefficient of inverse constraint with respect to position velocity (N) is of the same order as Γ.

Example 2. A gyroscope in universal joint has, also, a single position coordinate ϑ. Stabilization of steady motions belonging to leaves P_7, P_8 is in this case also possible. When $I_{2*}>I_{3*}$ it is then possible to set $\Gamma=0$, while for $I_{2*}<I_{3*}$ we can limit the choice to $\Gamma=kE$, where k is a scalar quantity. We have

$$W=(I_{2*}-I_{3*})\omega_2^2\sin^2\vartheta$$
$$W'=k[(I_{2*}-I_{3*})\omega_2\sin 2\vartheta-I_3\omega_3\sin\vartheta]^2+kI_3^2\omega_2^2\sin^2\vartheta$$

To satisfy the condition $W_0+W_0'>0$ it is sufficient to select k such that $kI_3^2>I_{3*}-I_{2*}$. Unlike in the previous example, it is possible to take the same value of the inverse constraint parameter k (and all others) for the stabilization of all significant steady motions.

4. On stability with weak control. In some applied problems it is expedient to analyze the effect of forces F_j with small coefficients

$$F_j=\varepsilon(K_j\eta+M_jx+N_jx^{\cdot})$$

where ε is a small parameter. When $\varepsilon=0$, the system of Eqs.(2.1) has the invariant manifold

$$W_0x-\left[\frac{\partial}{\partial q}(\omega^T B)\right]_0\eta=0 \tag{4.1}$$

which corresponds to zero roots of the characteristic equation. Let matrix W_0+W_0' be positive definite when $\Gamma = B^{-1}$. The manifold (4.1) is asymptotically stable. We denote by λ_m ($\lambda_m < 0$) the maximum real part of nonzero roots of the characteristic equation of system (2.1) for $\varepsilon = 0$.

The following analysis is based on the evident property of linear systems whose coefficients continuously depend on parameter ε. We separate some group of roots $\lambda_1(\varepsilon), \ldots, \lambda_k(\varepsilon)$ of the characteristic equation, and the corresponding to them invariant vectors $x^{(1)}(\varepsilon), \ldots, x^{(k)}(\varepsilon)$. Consider the parameter variation interval $(\varepsilon_1, \varepsilon_2)$ where none of the separated roots becomes multiple (although among the group of separated roots there may be some multiple ones). Then the invariant manifold generated by vectors $x^{(1)}(\varepsilon), \ldots, x^{(k)}(\varepsilon)$ continuously depends on ε for $\varepsilon \in (\varepsilon_1, \varepsilon_2)$ (although there may be no continuous dependence of vectors $x^{(1)}(\varepsilon), \ldots, x^{(k)}(\varepsilon)$ themselves).

The indicated property enables us to note that for $|\varepsilon| < \varepsilon_0$ (ε_0 is some positive quantity) system (2.1) has an invariant manifold of the form

$$W_0 x - \left[\frac{\partial}{\partial q}(\omega^T B)\right]_0 \eta + \mu(\varepsilon) S(\varepsilon)\eta = 0 \tag{4.2}$$

where S is some matrix, and as $\varepsilon \to 0$, the scalar $\mu(\varepsilon) \to 0$.

We carry out the substitution of time and variables η, by introducing

$$\tau = \varepsilon t, \quad y = B\eta + x^T \left[\frac{\partial}{\partial q}(B\omega)\right]_0$$

and represent system (2.1) in the manifold (4.2) as

$$\frac{dy}{d\tau} = \left\{KB^{-1} - MW_0^{-1}\left[\frac{\partial}{\partial q}(\omega^T B)\right]_0\right\} y + O(\mu) \tag{4.3}$$

where the last term contains all terms that vanish when $\mu = 0$.

The dimension of the system of Eqs.(4.3) is substantially lower than that of the input system and contains the small parameter μ, which enables us to apply to it known methods.

Let $\varkappa_m$ ($\varkappa_m \neq 0$) be the maximum real part of roots of the characteristic equation of system (4.3) with $\mu = 0$. The sign of $\varkappa_m$ obviously makes possible to judge on the stability or instability of the zero solution of system (2.1), provided the following two conditions are satisfied: $|\mu|$ is small in comparison with $|\varkappa_m|$ and $|\varepsilon\varkappa_m|$ is small in comparison with $|\lambda_m|$. The first of these ensures that the signs of $\varkappa_m$ and of the maximum real part of roots of the characteristic equation of system (4.3) are the same when the parameter μ is fairly small. Because of this it is possible to ascertain the instability of the zero solution (of steady motion) when $\varkappa_m > 0$.

When the second condition is satisfied, the invariant manifold (4.2) contains invariant vectors that correspond to roots of the characteristic equation of system (2.1) with the maximum real part, when the parameter ε is reasonably small. Both conditions are necessary for the validating the conclusion about the stability of the zero solution (stable motion) when $\varkappa_m < 0$.

The statement on the impossibility of stabilizing a steady motion with fairly small parameter ε when $\lambda_m > 0$ is also valid.

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